Fundamental Bounds In Electromagnetism

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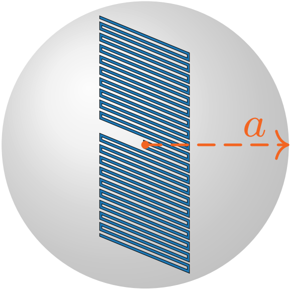
ČES Seminar, Prague, Czech Republic

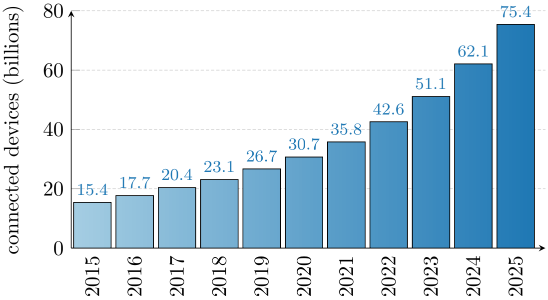
Outline



Electrically small antenna inside a circumscribing sphere of a radius a.

- Document available at capek.elmag.org.
- To see the graphics in motion, open this document in Adobe Reader!

Designing EM Devices...



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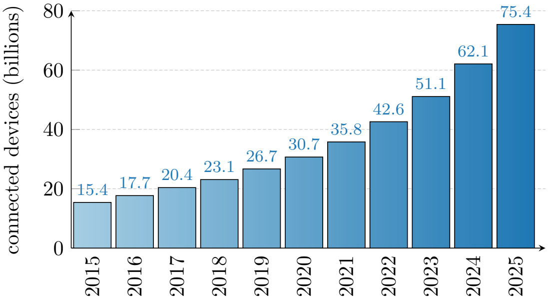
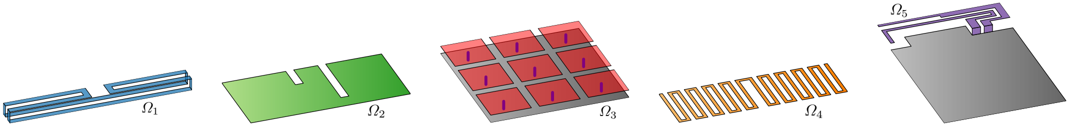


- What is the optimal design?



Folded loop (handsets) | E-shaped patch (GPS, WLAN) | "Mag. monopoles" (PGB, HIS) | Meandered dipole (RFID) | Monopoles/PIFAs (LTE)

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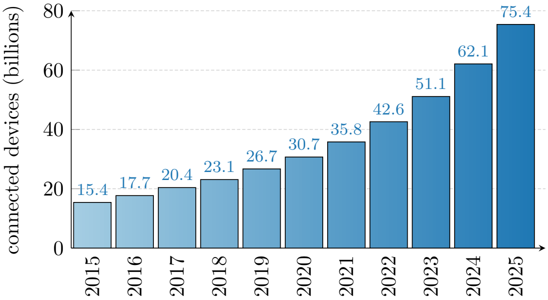

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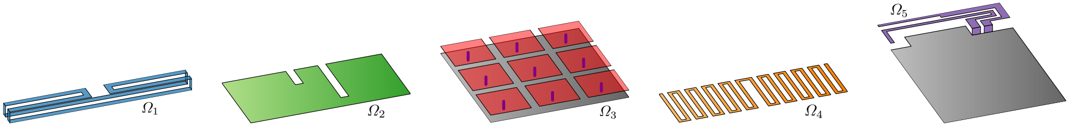
- Optimal design for what. . . ?



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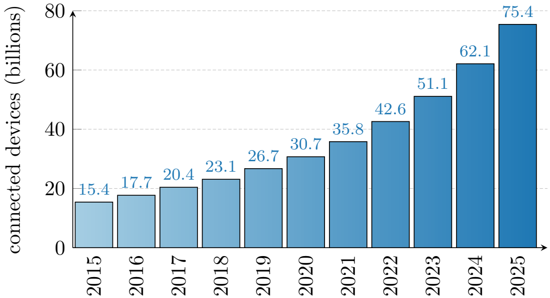
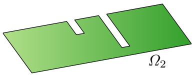


- What is the optimal design?
- Optimal design for what. . . ?
- What is the optimal performance?

Folded loop
(handsets)



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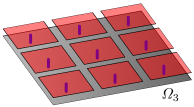

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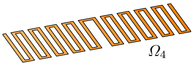

Meandered dipole
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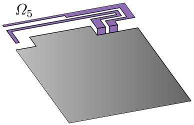
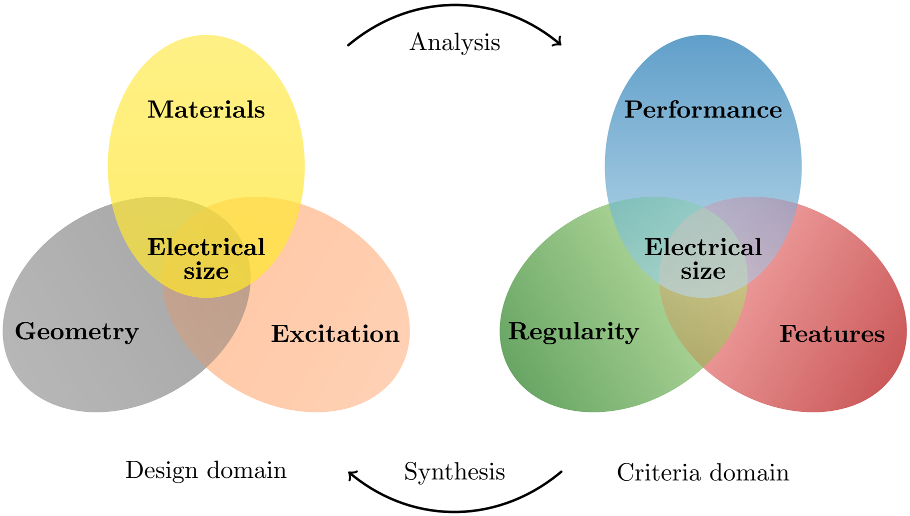
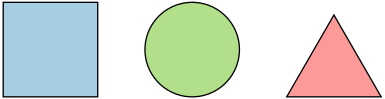


Monopoles/PIFAs
(LTE)

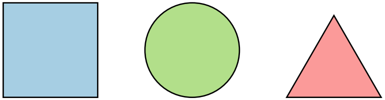
Degrees of Freedom and Figure of Merits

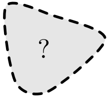




Analysis

- Shape is given, feeding is known.
- The task is to determine EM quantities.





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Synthesis (Inverse design)

- EM behavior is specified.
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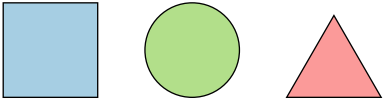

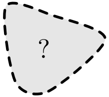

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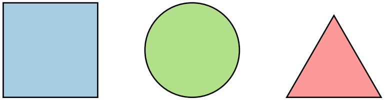

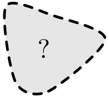

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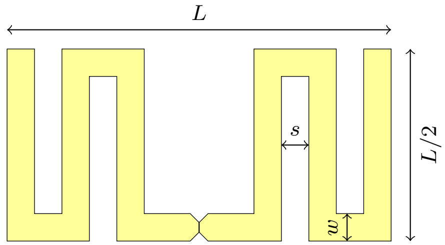
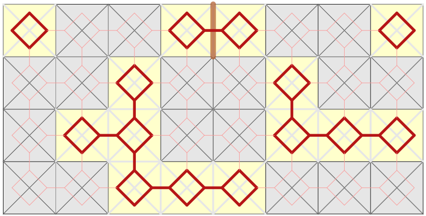
- Unsolved (except of rare cases).
- NP-hard/NP-complete.

Design Strategies

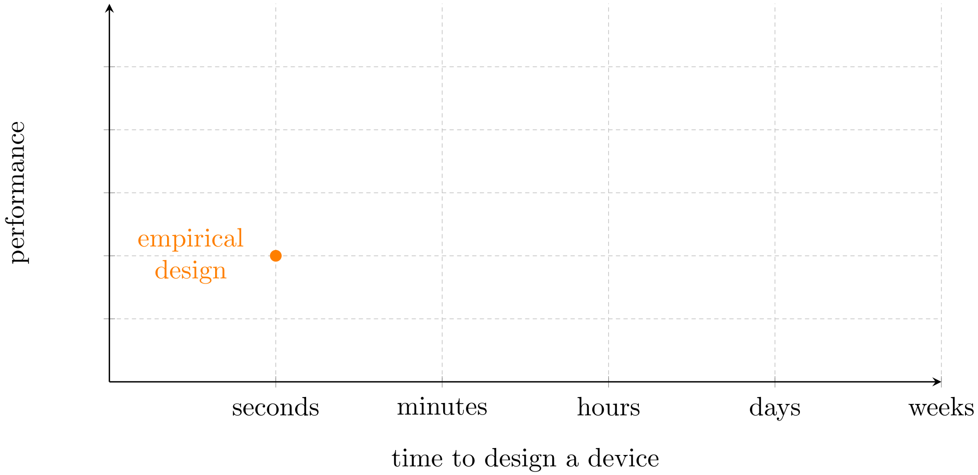
1. Designer's skill, experiences, and intuition.
2. Parameter sweep for predefined shapes.
3. Design libraries.
4. Local optimization (gradient-based).
5. Global optimization (heuristics).
6. Memetics, machine-learning-assisted techniques.





$i = 0,\ Q/Q_{\mathrm{lb}}^{\mathrm{TM}} = 119.35$

Design Curve



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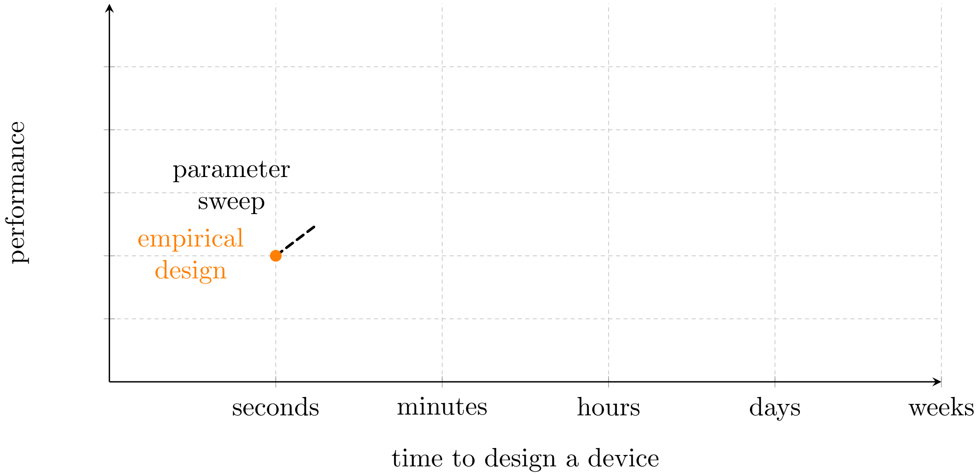

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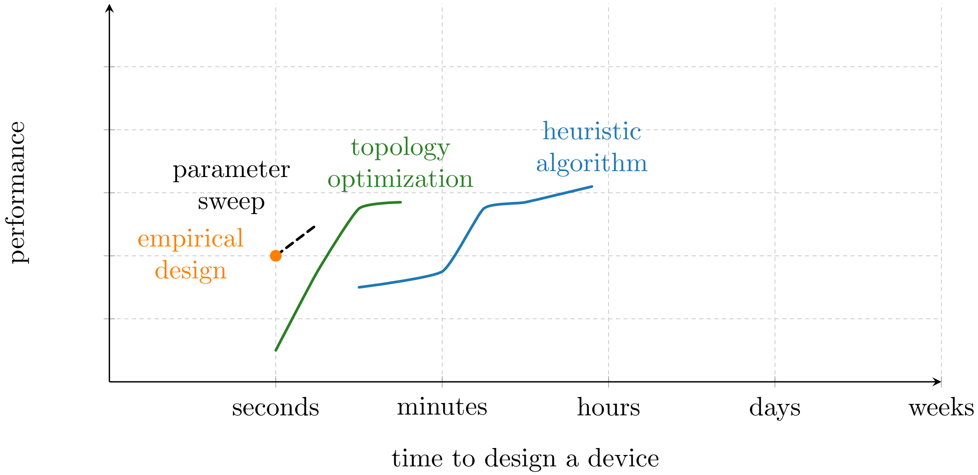

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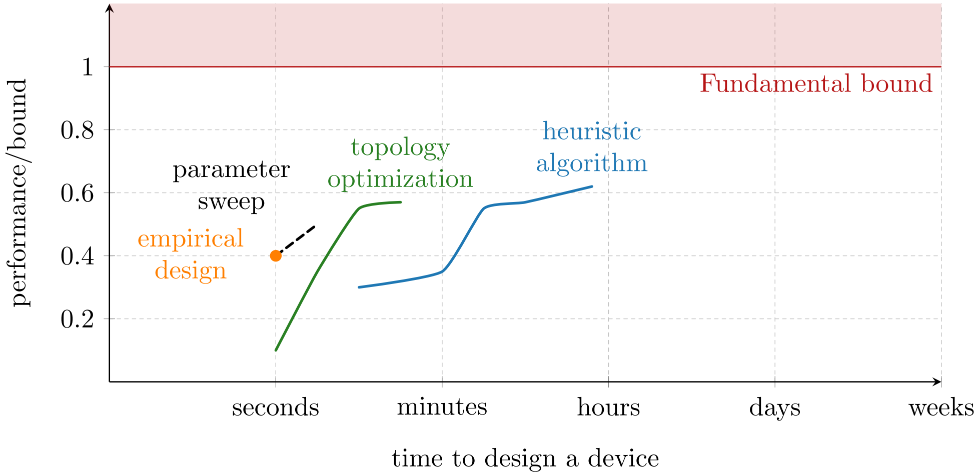

Fundamental Bounds

Example: Energy Extraction

Combustion

Fundamental Bounds

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Combustion

Nuclear fission

Fundamental Bounds

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Nuclear fusion

Fundamental Bounds

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$$\frac{W}{W_{\text{bound}}} \approx 10^{-9}$$

Combustion

$$\frac{W}{W_{\text{bound}}} \approx 10^{-3}$$

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$$\frac{W}{W_{\text{bound}}} \approx 10^{-2}$$

Nuclear fusion

What is the physical bound on energy production from fuel with mass m? $W_{\text{bound}} = mc^2$

Fundamental Bounds

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Combustion

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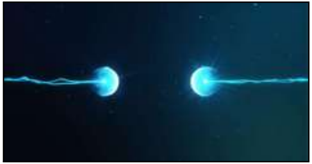
Nuclear fission

$$\frac{W}{W_{\text{bound}}} \approx 10^{-2}$$

Nuclear fusion

$$\frac{W}{W_{\text{bound}}} = 1$$

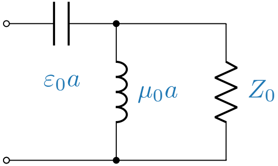


Annihilation of matter and antimatter

What is the physical bound on energy production from fuel with mass m? $W_{\text{bound}} = mc^2$

Approaching Fundamental Bounds in EM – Overview

- Circuit quantities (*e.g.*, equivalent circuits).
  - Wheeler (radiation power factor, 1947)
  - Chu (Q-factor, 1948)
  - Fano (matching, 1950)
  - Thal (Q-factor, 1978)
  - Pfeiffer (radiation efficiency, 2017)



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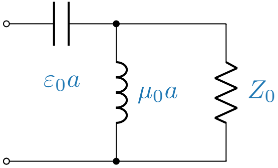

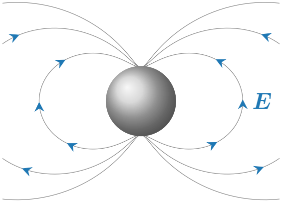

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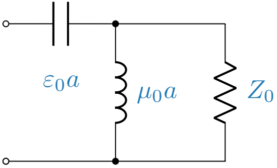

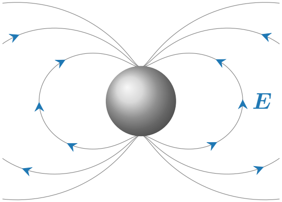

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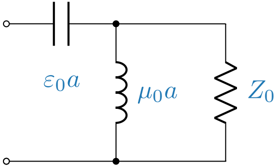

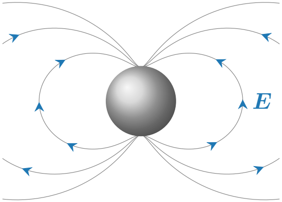

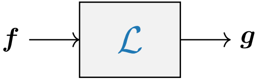

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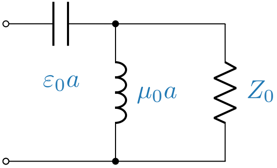

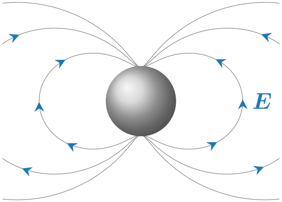

First Attempts: Directivity

What is the highest achievable directivity of an antenna?

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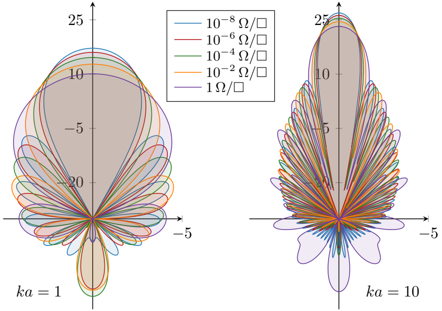
- It is possible to design an antenna of arbitrarily small dimensions with a directivity as high as desired[1].



[4]C. W. Oseen, “Die Einsteinsche Nadelstichstrahlung und die Maxwellschen Gleichungen,” *Ann. Phys.*, vol. 69, no. 19, pp. 202–204, 1922

First Attempts: Q-factor

What is the highest achievable fractional bandwidth[2] of a single-resonant antenna?

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$$\mathrm{FBW} < \frac{2\,|\Gamma|}{Q_{\mathrm{Chu}}} \qquad (1)$$

$$Q_{\mathrm{Chu}} = \frac{1}{2}\left(\frac{1}{(ka)^3} + \frac{2}{ka}\right) \qquad (2)$$

Key ingredient: Expansion of field into spherical waves.

[5] L. J. Chu, "Physical limitations of omni-directional antennas," *J. Appl. Phys.*, vol. 19, pp. 1163–1175, 1948

First Attempts: Away From Spheres

- Spherical waves are only suitable for spherical design regions.
- The developed bounds are relatively loose as compared to common antenna desings.

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"Shape-specific" fundamental bounds[3]

Given a specific design region, what is the best performace we can get from a device build in this region from a given material?

[5] M. Uzsoky and L. Solymár, "Theory of super-directive linear arrays," *Acta Physica Academiae Scientiarum Hungaricae*, vol. 6, no. 2, pp. 185–205, 1956
R. F. Harrington, "Antenna excitation for maximum gain," *IEEE Trans. Antennas Propag.*, vol. 13, no. 6, pp. 896–903, 1965

Example: Radiation Efficiency and Dissipation Factor

Radiation efficiency[4]:

$$\eta_{\mathrm{rad}} = \frac{P_{\mathrm{rad}}}{P_{\mathrm{rad}} + P_{\mathrm{lost}}} = \frac{1}{1 + \delta_{\mathrm{lost}}} \tag{3}$$

Dissipation factor[5] δ:

$$\delta_{\mathrm{lost}} = \frac{P_{\mathrm{lost}}}{P_{\mathrm{rad}}} \tag{4}$$

- fraction of quadratic forms (can be scaled with resistivity model).

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[5] R. F. Harrington, "Effect of antenna size on gain, bandwidth, and efficiency," *J. Res. Nat. Bur. Stand.*, vol. 64-D, pp. 1–12, 1960

Integral Operators and Their Algebraic Representation

Radiated and reactive power:

$$P_{\mathrm{rad}} + 2\mathrm{j}\omega \left(W_{\mathrm{m}} - W_{\mathrm{e}}\right) = \frac{1}{2} \langle \boldsymbol{J}\left(\boldsymbol{r}\right), \mathcal{Z}\left[\boldsymbol{J}\left(\boldsymbol{r}\right)\right]\rangle$$

Lost power (surface resistivity model):

$$P_{\mathrm{lost}} = \frac{1}{2} \langle \boldsymbol{J}\left(\boldsymbol{r}\right), \mathrm{Re}\left\{Z_{\mathrm{s}}\right\} \boldsymbol{J}\left(\boldsymbol{r}\right)\rangle$$

- The same approach as with the method of moments[6] (MoM)

$$\boldsymbol{J}\left(\boldsymbol{r}\right) \approx \sum_n I_n \boldsymbol{\psi}_n\left(\boldsymbol{r}\right)$$

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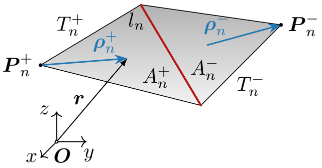

RWG basis function $\boldsymbol{\psi}_n$.

[6] R. F. Harrington, *Field Computation by Moment Methods*. Piscataway, New Jersey, United States: Wiley – IEEE Press, 1993

Algebraic Representation of Integral Operators

Radiated and reactive power

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Electric Field Integral Equation[7] (EFIE), $\mathbf{Z} = [Z_{mn}]$:

$$Z_{mn} = \int_{\Omega} \boldsymbol{\psi}_m \cdot \mathcal{Z}\left(\boldsymbol{\psi}_n\right) \mathrm{d}S = \mathrm{j}kZ_0 \int_{\Omega} \int_{\Omega} \boldsymbol{\psi}_m\left(\boldsymbol{r}_1\right) \cdot \mathbf{G}\left(\boldsymbol{r}_1, \boldsymbol{r}_2\right) \cdot \boldsymbol{\psi}_n\left(\boldsymbol{r}_2\right) \mathrm{d}S_1 \, \mathrm{d}S_2. \tag{6}$$

- Dense, symmetric matrix.
- An output from PEC 2D/3D MoM code (Ansys FEKO, CST MWS, HFSS,...).

Algebraic Representation of Integral Operators

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[7] W. C. Chew, M. S. Tong, and B. Hu, *Integral Equation Methods for Electromagnetic and Elastic Waves*. Morgan & Claypool, 2009

Algebraic Representation of Integral Operators

Radiated and reactive power

$$P_{\mathrm{rad}} + 2\mathrm{j}\omega\left(W_{\mathrm{m}} - W_{\mathrm{e}}\right) = \frac{1}{2}\left\langle \boldsymbol{J}\left(\boldsymbol{r}\right), \mathcal{Z}\left[\boldsymbol{J}\left(\boldsymbol{r}\right)\right]\right\rangle \approx \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathbf{Z}\mathbf{I} \tag{5}$$

Electric Field Integral Equation[7] (EFIE), $\mathbf{Z} = \left[Z_{mn}\right]$:

$$Z_{mn} = \int\limits_{\Omega} \boldsymbol{\psi}_m \cdot \mathcal{Z}\left(\boldsymbol{\psi}_n\right) \mathrm{d}S = \mathrm{j}kZ_0 \int\limits_{\Omega}\int\limits_{\Omega} \boldsymbol{\psi}_m\left(\boldsymbol{r}_1\right) \cdot \mathbf{G}\left(\boldsymbol{r}_1, \boldsymbol{r}_2\right) \cdot \boldsymbol{\psi}_n\left(\boldsymbol{r}_2\right) \mathrm{d}S_1\,\mathrm{d}S_2. \tag{6}$$

- Dense, symmetric matrix.
- An output from PEC 2D/3D MoM code (Ansys FEKO, CST MWS, HFSS,...).

[7] W. C. Chew, M. S. Tong, and B. Hu, *Integral Equation Methods for Electromagnetic and Elastic Waves*. Morgan & Claypool, 2009

Algebraic Representation of Integral Operators

Lost power

$$P_{\mathrm{lost}} = \frac{1}{2} \langle \boldsymbol{J}(\boldsymbol{r}), \mathrm{Re}\{Z_{\mathrm{s}}\} [\boldsymbol{J}(\boldsymbol{r})] \rangle \tag{7}$$

Algebraic Representation of Integral Operators

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$$L_{mn} = \int_{\Omega} \boldsymbol{\psi}_m \cdot \boldsymbol{\psi}_n \, \mathrm{d}S \tag{8}$$

Surface resistivity model:

$$Z_{\mathrm{s}} = \frac{1 + \mathrm{j}}{\sigma \delta} \tag{9}$$

with skin depth $\delta = \sqrt{2/\omega\mu_0\sigma}$.

Algebraic Representation of Integral Operators

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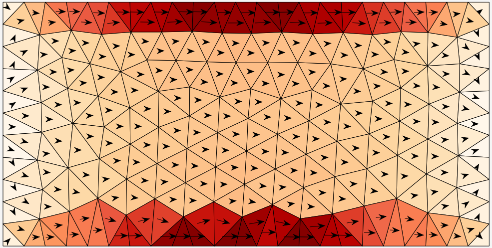
with skin depth $\delta = \sqrt{2/\omega\mu_0\sigma}$.

- Sparse matrix (diagonal for non-overlapping functions $\{\boldsymbol{\psi}_m(\boldsymbol{r})\}$).
- The entries L_{mn} are known analytically.

A Note: MoM Solution × Current Impressed in Vacuum

MoM solution



Solution to $\mathbf{I} = \mathbf{Z}^{-1}\mathbf{V}$ for an incident plane wave.

A current can be chosen completely freely, only the excitation $\mathbf{V} = \mathbf{Z}\mathbf{I}$ may not be realizable.

A Note: MoM Solution × Current Impressed in Vacuum

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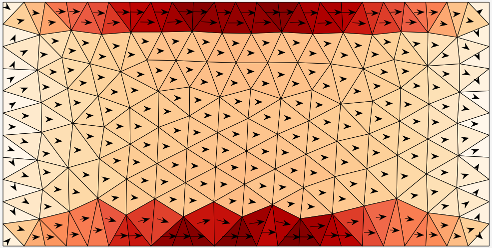

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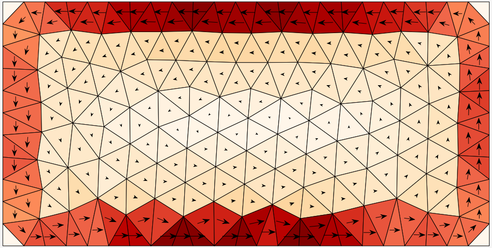

Solution to $\mathbf{X}\mathbf{I}_i = \lambda_i \mathbf{R}\mathbf{I}_i$ (the first inductive mode).

A current can be chosen completely freely, only the excitation $\mathbf{V} = \mathbf{Z}\mathbf{I}$ may not be realizable.

Fundamental Bounds as QCQP Problems

- Having quadratic forms for the physical quantities, the antenna metrics may be optimized.

Maximum radiation efficiency

Problem $\mathcal{P}_1$:

$$\begin{aligned} \text{minimize} \quad & P_{\mathrm{loss}} \\ \text{subject to} \quad & P_{\mathrm{rad}} = 1 \end{aligned}$$

Maximum self-resonant radiation efficiency

Problem $\mathcal{P}_2$:

$$\begin{aligned} \text{minimize} \quad & P_{\mathrm{loss}} \\ \text{subject to} \quad & P_{\mathrm{rad}} = 1 \\ & \omega \left(W_{\mathrm{m}} - W_{\mathrm{e}} \right) = 0 \end{aligned}$$

Fundamental Bounds as QCQP Problems

- Having quadratic forms for the physical quantities, the antenna metrics may be optimized.
- The problems $\mathcal{P}_1$ and $\mathcal{P}_2$ are quadratically constrained quadratic programs[8] (QCQP).

Maximum radiation efficiency

Problem $\mathcal{P}_1$:

$$\begin{aligned} \text{minimize} \quad & \mathbf{I}^{\mathrm{H}}\mathbf{L}\mathbf{I} \\ \text{subject to} \quad & \mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I} = 1 \end{aligned}$$

Maximum self-resonant radiation efficiency

Problem $\mathcal{P}_2$:

$$\begin{aligned} \text{minimize} \quad & \mathbf{I}^{\mathrm{H}}\mathbf{L}\mathbf{I} \\ \text{subject to} \quad & \mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I} = 1 \\ & \mathbf{I}^{\mathrm{H}}\mathbf{X}\mathbf{I} = 0 \end{aligned}$$

[8] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, Great Britain: Cambridge University Press, 2004

Solution to Radiation Efficiency Bound ($\mathcal{P}_1$)

Lagrangian reads

$$\mathcal{L}(\lambda, \mathbf{I}) = \mathbf{I}^{\mathrm{H}}\mathbf{L}\mathbf{I} - \lambda\left(\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I} - 1\right). \quad (10)$$

Stationary points

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I}^{\mathrm{H}}} = \mathbf{L}\mathbf{I} - \lambda\mathbf{R}\mathbf{I} = 0 \quad (11)$$

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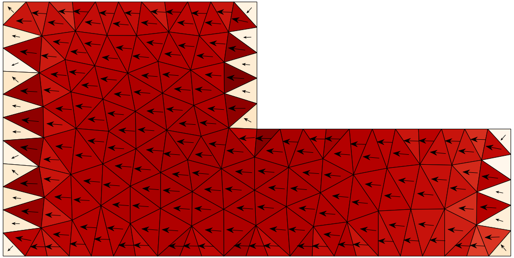
Example: Radiation Efficiency Bound of an L-plate ($\mathcal{P}_1$)

$ka = 1$, $R_\mathrm{s} = 0.01\,\Omega/\square$.



Optimal current (1st mode), $Z_0/R_\mathrm{s}\,(ka)^2\,\delta_\mathrm{loss} = 17.6$.

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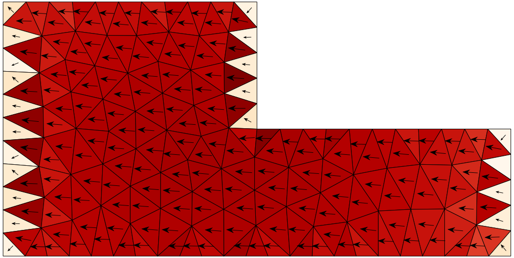

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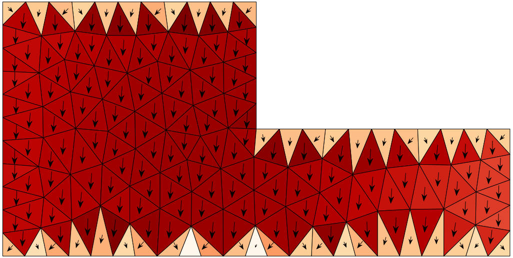

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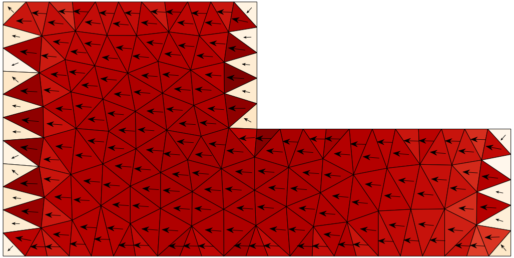

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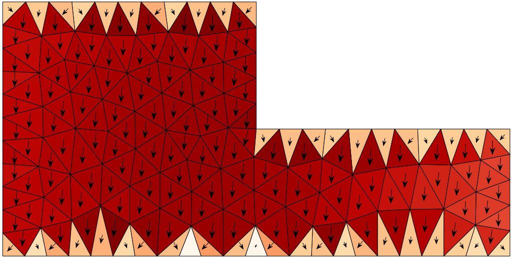

The 2nd current mode, $Z_0/R_\mathrm{s}\,(ka)^2\,\delta_\mathrm{loss} = 19.2$.

- Constant current has the lowest ohmic losses compared to its radiation.
- Clearly, such current is not realizable (and singular on the boundary).

Solution to Self-Resonant Radiation Efficiency Bound ($\mathcal{P}_2$)

The same solving procedure[9] as with problem $\mathcal{P}_1$, two Lagrange multipliers, however:

$$\mathcal{L}(\lambda_1, \lambda_2, \mathbf{I}) = \mathbf{I}^{\mathrm{H}}\mathbf{L}\mathbf{I} - \lambda_1 \left(\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I} - 1\right) - \lambda_2 \mathbf{I}^{\mathrm{H}}\mathbf{X}\mathbf{I}. \tag{14}$$

Stationary points

$$\left(\mathbf{L} - \lambda_2 \mathbf{X}\right) \mathbf{I}_i = \lambda_{1,i} \mathbf{R}\mathbf{I}_i. \tag{15}$$

[9] M. Gustafsson and M. Capek, "Maximum gain, effective area, and directivity," *IEEE Trans. Antennas Propag.*, vol. 67, no. 8, pp. 5282 –5293, 2019

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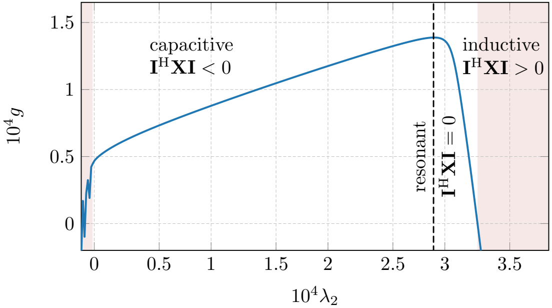

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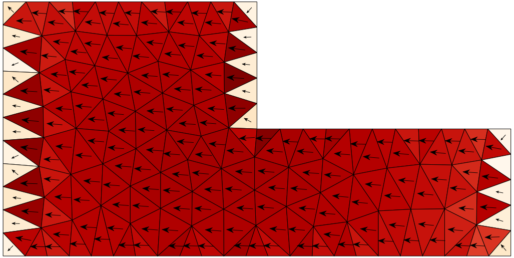
Example: Optimal Currents for L-Shape Plate ($\mathcal{P}_1$ & $\mathcal{P}_2$)

$ka = 1$, $R_\mathrm{s} = 0.01\,\Omega/\square$.



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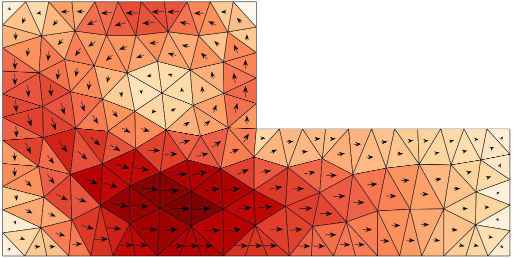

Optimal current for $\mathcal{P}_2$,
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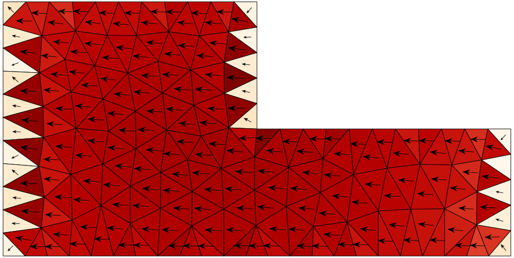

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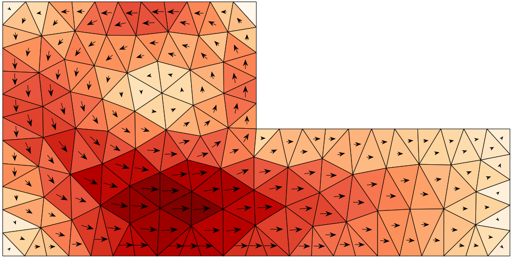

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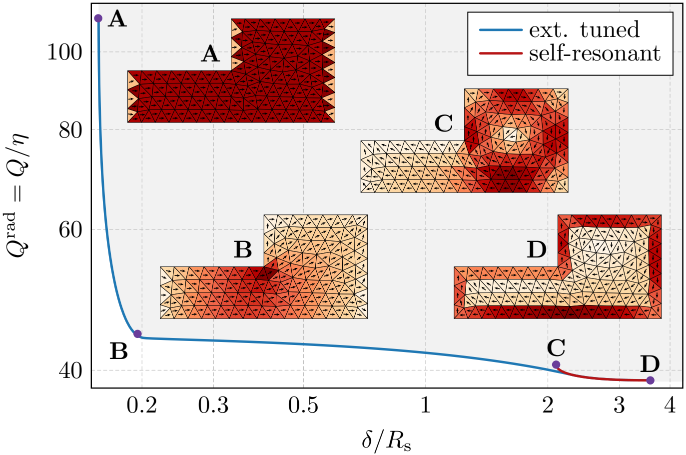
The same approach may be applied for any representation of the integral operators.

- Surface MoM, separable bodies, volumetric MoM, hybrid integral methods.

Trade-off Between Antenna Metrics

Example: Radiation efficiency vs. antenna bandwidth[10], $ka = 1/2$, $R_\mathrm{s} = 1\,\Omega/\square$



[10] M. Gustafsson, M. Capek, and K. Schab, "Tradeoff between antenna efficiency and Q-factor," *IEEE Trans. Antennas Propag.*, vol. 67, no. 4, pp. 2482–2493, 2019

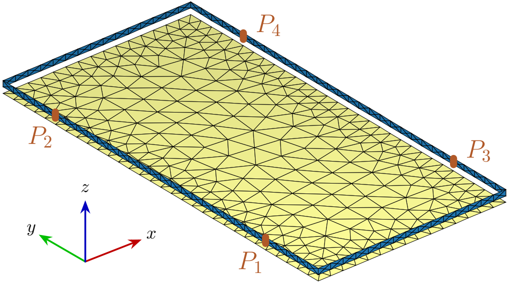
TARC Minimization

Total active reflection coefficient (TARC)

$$\Gamma^{\mathrm{t}} = \sqrt{1 - \frac{P_{\mathrm{rad}}}{P_{\mathrm{in}}}} = \sqrt{1 - \frac{\mathbf{v}^{\mathrm{H}}\mathbf{g}_0\mathbf{v}}{\mathbf{v}^{\mathrm{H}}\mathbf{k}_{\mathrm{i}}^{\mathrm{H}}\mathbf{k}_{\mathrm{i}}\mathbf{v}}} \qquad (16)$$

is to be minimized with QCQP[11]:

$$\begin{aligned} \text{maximize} \quad & \mathbf{v}^{\mathrm{H}}\mathbf{g}_0\mathbf{v} \\ \text{subject to} \quad & \mathbf{v}^{\mathrm{H}}\mathbf{k}_{\mathrm{i}}^{\mathrm{H}}\mathbf{k}_{\mathrm{i}}\mathbf{v} = 1 \end{aligned} \qquad (17)$$



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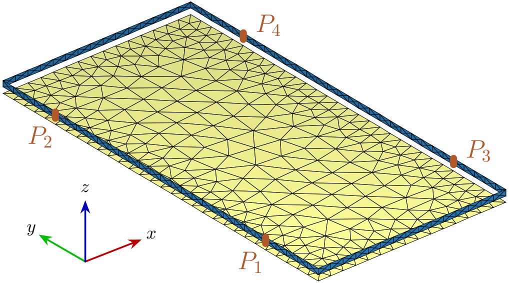

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Various levels of complexity:

- optimal excitation of ports,
- optimal placement of ports,

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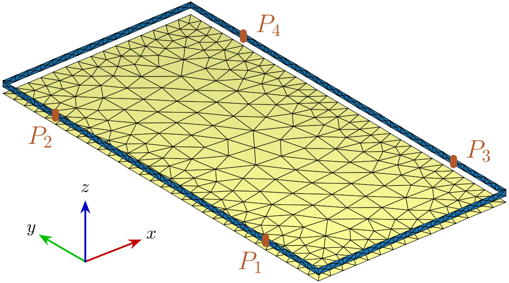

Various levels of complexity:

- optimal excitation of ports,
- optimal placement of ports,
- optimal number of ports,
- optimal matching circuitry.

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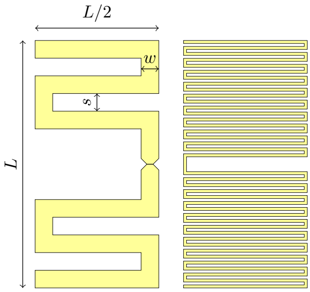
Shapes Known to Be Optimal (In Certain Sense)

Radiation Q-factor[12]



(a) (b)

Possible parametrization (unknowns: s, w, *i.e.*, number of meanders).

[12] M. Capek, L. Jelinek, K. Schab, *et al.*, "Optimal planar electric dipole antennas: Searching for antennas reaching the fundamental bounds on selected metrics," *IEEE Antennas and Propagation Magazine*, vol. 61, no. 4, pp. 19–29, 2019

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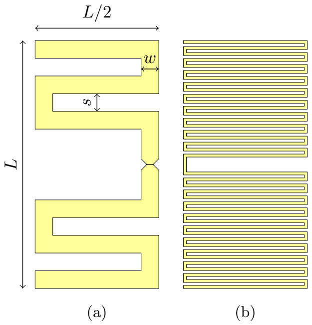

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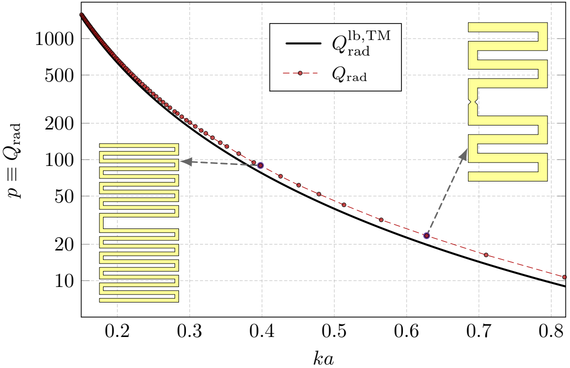
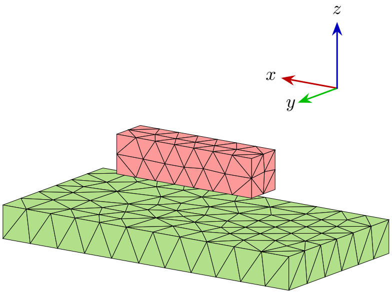


Q-factor of meanderline antennas compared to the bound.

[12] M. Capek, L. Jelinek, K. Schab, *et al.*, "Optimal planar electric dipole antennas: Searching for antennas reaching the fundamental bounds on selected metrics," *IEEE Antennas and Propagation Magazine*, vol. 61, no. 4, pp. 19–29, 2019

Shapes Known to Be Optimal (In Certain Sense)

Cloaking efficiency (extinction cross section)



A (fixed) rod over a slab (optimized).

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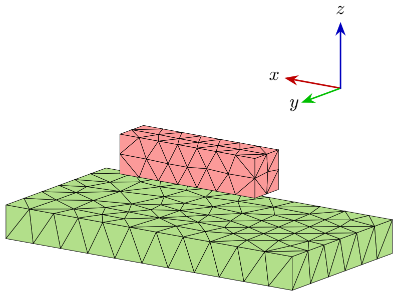

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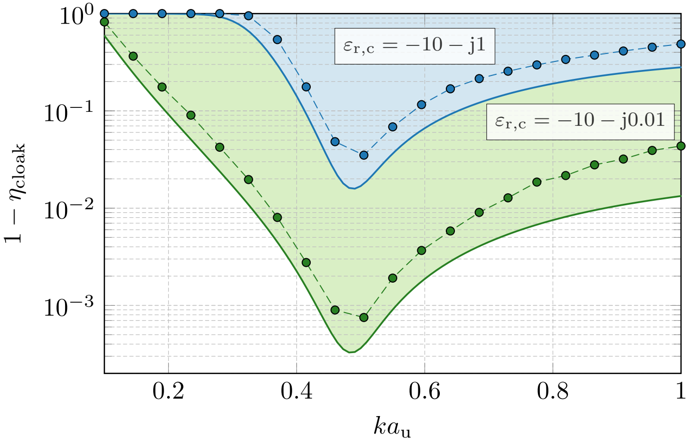


Cloaking efficiency of optimized slabs compared to the bound $\eta_{\mathrm{cloak}}^{\mathrm{ub}}$.

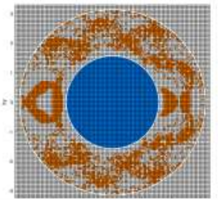
Conclusion

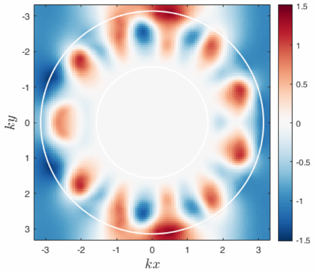
Bounds (QCQP)

- Help us to understand principal limits.
- We know when to stop with the design procedure.
- Applicable to arbitrarily shaped bodies.
- Inhomogeneous materials, combined metrics, trade-offs.
- Supports constraints on input impedance, complex power, directional constraints, polarization, etc.
- Sometimes directly realizable (port-modes).

Future

- Other metrics and their bounds.
- So far only single-frequency.
- Piecewise constraints (local power conservation).





Questions?

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ČES Seminar, Prague, Czech Republic
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